SUPEREXTENSION $n = (2,2)$ OF THE COMPLEX LIOUVILLE EQUATION AND ITS SOLUTION [a]

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It is shown that the method of the nonlinear realization of local supersymmetry previously developed in framework of supergravity being applied to the $n = (2,2)$ superconformal symmetry allows one to get the new form of the exactly solvable $n = (2,2)$ super-Liouville equation. The general advantage of this version as compared with the conventional one is that its bosonic part includes the complex Liouville equation. We obtain the suitable supercovariant constraints imposed on the corresponding superfields which provide the set of the resulting system of component equations be the same as that in model of $N = 2$, $D = 4$ Green-Schwarz superstring. The general solution of this system is derived from the corresponding solution of the bosonic string equation.

Introduction

It is well-known that a doubly supersymmetric generalization of the geometrical approach to superstring [1,2] leads in the case of $N = 2$, $D = 3$ Green-Schwarz superstring to the new version of the Liouville equation referring in literature as $n = (1,1)$ [3]. The latter as one can expected include the real Liouville equation in its bosonic part. The problem, however, arise when we try to extend this result on the case of $N = 2$, $D = 4$ superstring. It turns out that in this case the well-known form of the suitable super-Liouville equation [4] is not relevant in virtue of absence of the complex Liouville equation in the corresponding bosonic part. Thus, the equation proposed in [4] can not be applied for description of the $N = 2$, $D = 4$ Green-Schwarz superstring, which as one know is reduced to ordinary complex Liouville equation when neglecting by all the fermionic component fields.

In this paper we would like to propose the new version of the $n = (2,2)$ super-Liouville equation which appears to be agreement with the equations of motion of the $N = 2$, $D = 4$ Green-Schwarz superstring. Our approach is based on the method of the nonlinear realization of local supersymmetries developed by Ivanov and Kapustnikov in frame of supergravity [5]. It will be shown that when applied to the $n = (2,2)$ superconformal symmetry this method makes the possibility to impose the supercovariant constraints on the

[a] Talk given at the XIV-th Max Born Symposium, Karpacz, Poland, September 21-25, 1999

superfields in such a way that all the unphysical degrees of freedom occur in the original equation will appear removed from the residual set of the equation of motion. The latter amounts to the complex Liouville equation for the bosonic worldsheet variable $\tilde{u}(\tilde{\xi}^{++},\ \tilde{\xi}^{--})$ supplemented with two first order free equations $\tilde{\partial}_{--}\lambda^{+}(\tilde{\xi}^{++},\ \tilde{\xi}^{--}) = \tilde{\partial}_{++}\lambda^{-}(\tilde{\xi}^{++},\ \tilde{\xi}^{--}) = 0$ for the fermions of opposite chirality.

In Section 3 we present the general solution of this equation in terms of the restricted Lorentz harmonic variables [6], which by a proper fashion extends a corresponding bosonic string solution obtained in [7].

1 New version of the $n = (2,2)$ super-Liouville equation

1.1 Linear realization

We begin with the linear realization of two copies of one dimensional superconformal group acting separately on the light-cone complex coordinates of $N = 2$, $D = 4$ superstring $\mathbf{C}^{(2|2)} = (\xi_L^{++} = \xi^{++} + i\eta^{+}\overline{\eta}^{+}, \eta^{+}; \xi_L^{--} = \xi^{--} + i\eta^{-}\overline{\eta}^{-}, \eta^{-})$:

$$\begin{aligned} \xi_L^{\pm\pm\prime} &= \Lambda^{\pm\pm} - \overline{\eta}^{\pm}\overline{D}_{\pm}\Lambda^{\pm\pm} \\ &= a_L^{\pm\pm}(\xi_L^{\pm\pm}) + 2i\eta^{\pm}\overline{\epsilon}^{\pm}(\xi_L^{\pm\pm})g^{(\pm\pm)}(\xi_L^{\pm\pm})e^{i\rho^{(\pm\pm)}(\xi_L^{\pm\pm})}, \\ \eta^{\pm\prime} &= -\frac{i}{2}\overline{D}_{\pm}\Lambda^{\pm\pm} = \epsilon^{\pm}(\xi_L^{\pm\pm}) + \eta^{\pm}g^{(\pm\pm)}(\xi_L^{\pm\pm})e^{i\rho^{(\pm\pm)}(\xi_L^{\pm\pm})}, \\ a_L^{\pm\pm}(\xi_L^{\pm\pm}) &= \xi_L^{\pm\pm} + a^{\pm\pm}(\xi_L^{\pm\pm}) + i\epsilon^{\pm}(\xi_L^{\pm\pm})\overline{\epsilon}^{\pm}(\xi_L^{\pm\pm}), \\ g^{(\pm\pm)} &= \sqrt{1 + \partial_{\pm\pm}a^{\pm\pm} + i(\epsilon^{\pm}\partial_{\pm\pm}\overline{\epsilon}^{\pm} + \overline{\epsilon}^{\pm}\partial_{\pm\pm}\epsilon^{\pm})}. \end{aligned} \tag{1}$$

In Eq. (1) the general superfield (SF)

$$\begin{aligned} \Lambda^{\pm\pm}(\xi_L^{\pm\pm}, \eta^{\pm}, \overline{\eta}^{\pm}) &= a_L^{\pm\pm}(\xi_L^{\pm\pm}) + 2i\eta^{\pm}\overline{\epsilon}^{\pm}(\xi_L^{++})g^{(\pm\pm)}(\xi_L^{\pm\pm})e^{i\rho^{(\pm\pm)}(\xi_L^{++})} \\ &+ 2i\overline{\eta}^{\pm}\epsilon^{\pm}(\xi_L^{\pm\pm}) - 2i\eta^{\pm}\overline{\eta}^{\pm}g^{(\pm\pm)}(\xi_L^{\pm\pm})e^{i\rho^{(\pm\pm)}(\xi_L^{\pm\pm})}, \end{aligned} \tag{2}$$

is composed out from parameters $\epsilon^{+}(\xi_L^{++})$, $\epsilon^{-}(\xi_L^{--})$ of local supertranslations; two real parameters $a^{++}(\xi_L^{++})$, $a^{--}(\xi_L^{--})$ of $D1$-reparametrizations and two real parameters $\rho^{++}(\xi_L^{++})$, $\rho^{--}(\xi_L^{--})$ describing local $U(1) \times U(1)$-rotations. The spinor covariant derivatives are defined as

$$\begin{aligned} D_{\pm} &= \partial_{\pm} + 2i\overline{\eta}^{\pm}\partial_{\pm\pm}, \\ \overline{D}_{\pm} &= \overline{\partial}_{\pm}. \end{aligned} \tag{3}$$

It is worth to mention that since the parameters $\xi_L^{\pm\pm\prime}$ and $\eta^{\pm\prime}$ in Eqs. (1) are subjected to the constraints

$$D_{\pm}\xi_L^{\pm\pm\prime} - 2i\overline{\eta}^{\pm\prime}D_{\pm}\eta^{\pm\prime} = 0 \tag{4}$$

the *flat* spinor covariant derivatives (3) are transformed homogeneously with respect to (1)

$$D_{\pm} = (D_{\pm}\eta^{\pm\prime})D'_{\pm}. \tag{5}$$

Therefore, the following superconformal-covariant equation can be proposed as a natural candidate for $n=(2,2)$ superextension of the corresponding $n=(1,1)$ super-Liouville equation [3]

$$D_{-}D_{+}W = e^{2W}\Psi_{+}^{--}\Psi_{-}^{++}. \tag{6}$$

In Eq. (6) one double-analytical SF

$$W(\xi_L^{\pm\pm},\eta^{\pm}) = u(\xi_L^{\pm\pm}) + \eta^{+}\psi^{-}(\xi_L^{\pm\pm}) + \eta^{-}\psi^{+}(\xi_L^{\pm\pm}) + \eta^{-}\eta^{+}F(\xi_L^{\pm\pm}), \tag{7}$$

and two general SFs $\Psi_{+}(\xi_L^{++},\eta^{+},\overline{\eta}^{+})$, $\Psi_{-}(\xi_L^{--},\eta^{-},\overline{\eta}^{-})$, depending separately on the $(2,0)$ and $(0,2)$ light-cone variables, are introduced. [b] Eq. (6) is invariant under the following gauge transformations

$$\begin{aligned}
W'(\xi_L^{\pm\pm\prime},\eta^{\pm\prime}) &= W(\xi_L^{\pm\pm},\eta^{\pm}) - \frac{1}{2}ln(\overline{D}_{+}\overline{\eta}^{+\prime}) - \frac{1}{2}ln(\overline{D}_{-}\overline{\eta}^{-\prime}), \\
\Psi'_{+}(\xi_L^{++\prime},\eta^{+\prime},\overline{\eta}^{+\prime}) &= (D_{+}\eta^{+\prime})^{-1}(\overline{D}_{+}\overline{\eta}^{+\prime})\Psi_{+}(\xi_L^{++},\eta^{+},\overline{\eta}^{+}), \\
\Psi'_{-}(\xi_L^{--\prime},\eta^{-\prime},\overline{\eta}^{-\prime}) &= (D_{-}\eta^{-\prime})^{-1}(\overline{D}_{-}\overline{\eta}^{-\prime})\Psi_{-}(\xi_L^{--},\eta^{-},\overline{\eta}^{-}).
\end{aligned} \tag{8}$$

Note that due to the nilpotence of the covariant derivatives ($D_{\pm}^2=0$) the SFs W and $\Psi_{\pm}$ included in the Eq. (6) appear restricted

$$D_{\pm}\Psi_{\pm} + 2(D_{\pm}W)\Psi_{\pm} = 0. \tag{9}$$

A particular property we shall encounter with here is, however, that the constraints (9) can be solved explicitly in terms of the *unrestricted* Fs

$$\Psi_{+} = D_{+}M + 2(D_{+}W)M, \qquad \Psi_{-} = D_{-}N + 2(D_{-}W)N. \tag{10}$$

In Eq. (10) the general SFs [c]

$$\begin{aligned}
M(\xi_L^{++},\eta^{+},\overline{\eta}^{+}) &= f(\xi_L^{++}) + \eta^{+}\omega^{-}(\xi_L^{++}) + \\
&\quad \overline{\eta}^{+}\overline{\chi}^{-}(\xi_L^{++}) + \eta^{+}\overline{\eta}^{+}m^{--}(\xi_L^{++}), \\
N(\xi_L^{--},\eta^{-},\overline{\eta}^{-}) &= g(\xi_L^{--}) + \eta^{-}\omega^{+}(\xi_L^{--}) + \\
&\quad \overline{\eta}^{-}\overline{\chi}^{+}(\xi_L^{--}) + \eta^{-}\overline{\eta}^{-}n^{++}(\xi_L^{--}),
\end{aligned} \tag{11}$$

[b] We omit temporarily the upper indices of SFs Ψ and M for the enlightening of formulas but we shall come back to them in Section 3.

[c] It can be shown that in the case of chiral SFs $M(\xi_L^{++},\eta^{+}), N(\xi_L^{--},\eta^{-})$ Eq. (6) is reduced to free one $D_{+}D_{-}\tilde{W}=0$ for the SF $\tilde{W} = W + \frac{1}{2}ln(MN)$.

are supposed transform as a superconformal densities

$$M'(\xi_L^{++\prime},\eta^{+\prime},\overline{\eta}^{+\prime}) = (\overline{D}_+\overline{\eta}^{+\prime})M(\xi_L^{++},\eta^+,\overline{\eta}^+),$$
$$N'(\xi_L^{--\prime},\eta^{-\prime},\overline{\eta}^{-\prime}) = (\overline{D}_-\overline{\eta}^{-\prime})N(\xi_L^{--},\eta^-,\overline{\eta}^-). \tag{12}$$

Although the component content of SFs W, M, N even upon the gauge fixing is still too large to be related with the $N = 2$, $D = 4$ superstring there is very important feature of Eq. (6). It contains *complex* Liouville equation in its bosonic part

$$\partial_{++}\partial_{--}u(\xi_L^{\pm\pm}) = \frac{1}{4}e^{2u(\xi_L^{\pm\pm})}m^{--}(\xi_L^{++})n^{++}(\xi_L^{--}) + ..., \tag{13}$$

where all the unessential terms in the r.h.s. are omitted. It is clear, however, that to be connected with the superstring theory the SFs we have considered here must be covariantly constrained. In the next Section we are going to show that the desirable constraints could be imposed in frame of the nonlinear realization of $n = (2, 2)$ superconformal symmetry in which the original SFs becomes reducible.

1.2 Nonlinear realization

To see this let us suppose that the v.e.v. of the component fields $m^{--}(\xi_L^{++})$ and $n^{++}(\xi_L^{--})$ in (11) are not equal to zero and as consequence of this the local supersymmetry (1) is actually spontaneously broken. In this case the fermionic components $\chi^\pm$ acquire the sense of the corresponding Goldstone fermions and one can exploit them for the singling out of the complex Liouville equation from the system (6) in a manifestly covariant manner. Indeed, it is well-known that in the models with spontaneously broken supersymmetry all the SFs becomes reducible [10], [5]. Their irreducible parts are transformed, however, universally with respect to the action of the original supergroups, as the linear representations of the underlying unbroken subgroups but with the parameters depending nonlinearly on the Goldstone fermions. It makes the possibility to impose generally on the SFs in question some absolutely covariant restrictions providing to remove out from the model under consideration undesirable degrees of freedom. Here we can to avail oneself of the opportunity to restrict the SFs enter the Eq. (6) with the help of this approach.

For the beginning let us derive the nonlinear realization of the superconformal symmetry in superspace. Following closely to the general method developed in [8], [5] we need firstly splits the general finite element of the group (1)

$$G(\zeta_L) \equiv \zeta_L', \tag{14}$$

where $\zeta_L = \{\xi_L^{\pm\pm}, \eta^\pm\}$, onto the product of two successive transformations

$$G(\zeta_L) = K(G_0(\zeta_L)). \tag{15}$$

In Eq. (15) the following standard notations are used. As before the $G_0(\zeta_L)$ refer to the "primes" coordinates ζ_L' but index *zero* means that they referring now only to the stability subgroup

$$\begin{aligned} \xi_L^{\pm\pm\prime} &= \xi_L^{\pm\pm} + a^{\pm\pm}(\xi_L^{\pm\pm}), \\ \eta^{\pm\prime} &= \eta^\pm e^{i\rho^{(\pm\pm)}(\xi_L^{\pm\pm})}\sqrt{1 + \partial_{\pm\pm} a^{\pm\pm}}. \end{aligned} \tag{16}$$

The latter include only the ordinary conformal transformations (parameters $a^{\pm\pm}(\xi_L^{\pm\pm})$) supplemented with the local $U(1) \times U(1)$-rotations (parameters $\rho^{(\pm\pm)}(\xi_L^{\pm\pm})$). Note, that the first multiplier in the decomposition (15) is easily recognized as the representatives of the left coset space G/G_0 [d]

$$\begin{aligned} K^{\pm\pm}(\zeta_L) &= \xi_L^{\pm\pm} + i\epsilon^\pm(\xi_L^{\pm\pm})\overline{\epsilon}^\pm(\xi_L^{\pm\pm}) \\ &\quad + 2i\eta^\pm\overline{\epsilon}^\pm(\xi_L^{\pm\pm})\sqrt{1 + i(\epsilon^\pm\partial_{\pm\pm}\overline{\epsilon}^\pm + \overline{\epsilon}^\pm\partial_{\pm\pm}\epsilon^\pm)}, \\ K^\pm(\zeta_L) &= \epsilon^\pm(\xi_L^{\pm\pm}) + \eta^\pm\sqrt{1 + i(\epsilon^\pm\partial_{\pm\pm}\overline{\epsilon}^\pm + \overline{\epsilon}^\pm\partial_{\pm\pm}\epsilon^\pm)}. \end{aligned} \tag{17}$$

It deserves to mention that in the decomposition (15) the comultipliers K and G_0 are chosen in such a way that the irreduciblity constraint (3) is satisfied separately for both of them. The prescription for constructing the corresponding nonlinear realization is as follows [5]. Let us identify the local parameters $\epsilon^\pm(\xi_L^{\pm\pm})$, $\overline{\epsilon}^\pm(\xi_L^{\pm\pm})$ in (17) with the Goldstone fields $\lambda^\pm(\xi_L^{\pm\pm})$, $\overline{\lambda}^\pm(\xi_L^{\pm\pm})$

$$\begin{aligned} \tilde{K}^{\pm\pm}(\tilde{\zeta}_L) &= \tilde{\xi}_L^{\pm\pm} + i\lambda^\pm(\tilde{\xi}_L^{\pm\pm})\overline{\lambda}^\pm(\tilde{\xi}_L^{\pm\pm}) \\ &\quad + 2i\tilde{\eta}^\pm\overline{\lambda}^\pm(\tilde{\xi}_L^{\pm\pm})\sqrt{1 + i(\lambda^\pm\tilde{\partial}_{\pm\pm}\overline{\lambda}^\pm + \overline{\lambda}^\pm\tilde{\partial}_{\pm\pm}\lambda^\pm)}, \\ \tilde{K}^\pm(\tilde{\zeta}_L) &= \lambda^\pm(\tilde{\xi}_L^{\pm\pm}) + \tilde{\eta}^\pm\sqrt{1 + i(\lambda^\pm\tilde{\partial}_{\pm\pm}\overline{\lambda}^\pm + \overline{\lambda}^\pm\tilde{\partial}_{\pm\pm}\lambda^\pm)} \end{aligned} \tag{18}$$

and take for $\tilde{K}(\tilde{\zeta}_L)$ the transformation law associated to (15)

$$G(\tilde{K}(\tilde{\zeta}_L)) = \tilde{K}'(\tilde{G}_0(\tilde{\zeta}_L)). \tag{19}$$

In Eq. (19) the newly introduced coordinates $\tilde{\zeta}_L = \{\tilde{\xi}_L^{\pm\pm}, \tilde{\eta}^\pm\}$ are transformed differently as compared with $\zeta_L = \{\xi_L^{\pm\pm}, \eta^\pm\}$ in (1). Indeed, in accordance

[d] In virtue of (15) all the parameters in (1) should be regarded as composite ones which are composed out from the parameters of transformations (16) and (17).

with (16) they change only under the vacuum stability subgroup

$$\tilde{\xi}_L^{--\prime} = \tilde{\xi}_L^{\pm\pm} + \tilde{a}^{\pm\pm}(\tilde{\xi}_L^{\pm\pm}), \qquad (20)$$
$$\tilde{\eta}^{\pm\prime} = \tilde{\eta}^{\pm} e^{i\tilde{\rho}^{(\pm\pm)}(\tilde{\xi}_L^{\pm\pm})} \sqrt{1 + \tilde{\partial}_{\pm\pm}\tilde{a}^{\pm\pm}},$$

where the parameters $\tilde{a}^{\pm\pm}(\tilde{\xi}_L^{\pm\pm})$ and $\tilde{\rho}^{(\pm\pm)}(\tilde{\xi}_L^{\pm\pm})$ turns out to be dependent nonlinearly on the fields $\lambda^{\pm}(\xi_L^{\pm\pm})$, $\overline{\lambda}^{\pm}(\xi_L^{\pm\pm})$. Eqs. (19) and (20) determine the transformation properties of the Goldstone fermions $\lambda^{\pm}(\xi_L^{\pm\pm})$, $\overline{\lambda}^{\pm}(\xi_L^{\pm\pm})$ with respect to the nonlinear realization of the superconformal group G in coset space (18).

2 Splitting superspace and irreducible form of SFs

Up to now we have dealt with only formal prescription of construction of the nonlinear realization of superconformal group G without any relation of this procedure to the original equation (6). Nevertheless, there is the simple possibility to gain a more deeper insight into the model we started with if we compare two Eqs. (14) and (19). We find that $\tilde{K}(\tilde{\zeta}_L)$ transform under G in precisely the same manner as the initial coordinates ζ_L of superspace $\mathbf{C}^{(2|2)}$. Thus we have the unique possibility to identify them

$$\zeta_L = \tilde{K}(\tilde{\zeta}_L). \qquad (21)$$

Eq. (21) establish the relationship between two forms of the realization of superconformal symmetries in superspace. One of the remarkable futures of the transformations (21) is that superspace of the nonlinear realization $\tilde{\mathbf{C}}^{(2|2)} = \tilde{\zeta}_L$ turns out to be completely "splitting" in virtue of the transformations (20) which are not mixed the bosonic and fermionic variables. Due to this very important fact the SFs of the nonlinear realization becomes actually reducible. Indeed, let us perform the change of variables (21) in the Eq. (6)

$$\tilde{D}_-\tilde{D}_+\tilde{W} = e^{2\tilde{W}}\tilde{\Psi}_+\tilde{\Psi}_-, \qquad (22)$$

where the SFs and covariant derivatives of the nonlinear realization (19), (20) and (21) are introduced

$$W = \tilde{W} - \frac{1}{2}ln(\overline{\tilde{D}}_+\overline{\eta}^+) - \frac{1}{2}ln(\overline{\tilde{D}}_-\overline{\eta}^-), \qquad D_{\pm} = (\tilde{D}_{\pm}\eta^{\pm})^{-1}\tilde{D}_{\pm}, \qquad (23)$$

$$\tilde{\Psi}_+ = \tilde{D}_+\tilde{M} + 2(\tilde{D}_+\tilde{W})\tilde{M}, \qquad \tilde{\Psi}_- = \tilde{D}_-\tilde{N} + 2(\tilde{D}_-\tilde{W})\tilde{N}, \qquad (24)$$
$$M(\xi_L^{++}, \eta^+, \overline{\eta}^+) = (\overline{\tilde{D}}_+\overline{\eta}^+)\tilde{M}(\tilde{\xi}_L^{++}, \tilde{\eta}^+, \overline{\tilde{\eta}}^+),$$

$$N(\xi_L^{--},\eta^-,\overline{\eta}^-) = (\overline{\tilde{D}}_-\overline{\eta}^-)\tilde{N}(\tilde{\xi}_L^{--},\tilde{\eta}^-,\overline{\tilde{\eta}}^-). \quad (25)$$

Note should be taken that the covariant derivatives $\tilde{D}_\pm$ in (22) have the same structure as those of linear realization (3). This follows from the structure of the coset space representatives (17) which are defined in such a way that the irreducibility conditions (4) are fulfilled for them automatically.

Although the form of the Eq. (22) is precisely the same as the original one (6) the SFs of the nonlinear realization appearing in (22) are distinguished drastically from the SFs of linear realization. As it follows from (20) and (5) the SFs $\tilde{W}$ and $\tilde{\Psi}$ are transformed under the action of G only with respect to their stability subgroup (20)

$$\begin{aligned}
\tilde{W}'(\tilde{\xi}_L^{--\prime},\tilde{\eta}^{\pm\prime}) &= \tilde{W}(\tilde{\xi}_L^{\pm\pm},\tilde{\eta}^{\pm}) - \frac{1}{2}ln(\overline{\tilde{D}}_+\overline{\tilde{\eta}}^{+\prime}) - \frac{1}{2}ln(\overline{\tilde{D}}_-\overline{\tilde{\eta}}^{-\prime}), \quad (26)\\
\tilde{M}'(\tilde{\xi}_L^{++\prime},\tilde{\eta}^{+\prime},\overline{\tilde{\eta}}^{+\prime}) &= (\overline{\tilde{D}}_+\overline{\tilde{\eta}}^{+\prime})\tilde{M}(\tilde{\xi}_L^{++},\tilde{\eta}^+,\overline{\tilde{\eta}}^+),\\
\tilde{N}'(\tilde{\xi}_L^{--\prime},\tilde{\eta}^{-\prime},\overline{\tilde{\eta}}^{-\prime}) &= (\overline{\tilde{D}}_-\overline{\tilde{\eta}}^{-\prime})\tilde{N}(\tilde{\xi}_L^{--},\tilde{\eta}^-,\overline{\tilde{\eta}}^-).
\end{aligned}$$

Substituting here the explicit form of gauge parameters deduced from the transformations (20)

$$\overline{\tilde{D}}_\pm\overline{\tilde{\eta}}^{\pm\prime} = e^{i\tilde{\rho}^{(\pm\pm)}(\tilde{\xi}_L^{\pm\pm})}\sqrt{1+\tilde{\partial}_{\pm\pm}\tilde{a}^{\pm\pm}(\tilde{\xi}_L^{\pm\pm})}, \quad (27)$$

one concludes that all the component fields of the SFs $\tilde{W}$ and $\tilde{M},\tilde{N}$ are transformed *independently* of each other. Thus we can put down the following manifestly covariant constraints

$$\tilde{W}(\tilde{\xi}_L^{\pm\pm},\tilde{\eta}^\pm) = \tilde{u}(\tilde{\xi}_L^{\pm\pm}), \quad (28)$$

$$\begin{aligned}
\tilde{M}(\tilde{\xi}_L^{++},\tilde{\eta}^+,\overline{\tilde{\eta}}^+) &= \tilde{\eta}^+\overline{\tilde{\eta}}^+\tilde{m}^{--}(\tilde{\xi}_L^{++}),\\
\tilde{N}(\tilde{\xi}_L^{--},\tilde{\eta}^-,\overline{\tilde{\eta}}^-) &= \tilde{\eta}^-\overline{\tilde{\eta}}^-\tilde{n}^{++}(\tilde{\xi}_L^{--}). \quad (29)
\end{aligned}$$

which leaves intact the G-invariance of theory. Returning these constraints back into the Eq. (22) we obtain the final component form of the Eq. (6)

$$\tilde{\partial}_{--}\tilde{\partial}_{++}\tilde{u} = \frac{1}{4}e^{2\tilde{u}}\tilde{m}^{--}(\tilde{\xi}_L^{++})\tilde{n}^{++}(\tilde{\xi}_L^{--}). \quad (30)$$

This Eq. together with chirality conditions of the Goldstone fermions $\lambda^\pm(\xi_L^{\pm\pm})$, $\overline{\lambda}^\pm(\xi_L^{\pm\pm})$ gives the whole system of Eqs. describing dynamics of the $N=2$, $D=4$ superstring in the component level [9].

3 General solution

Let us consider shortly the problem of construction of general solution of the Eq. (6). It is well-known that the Virasoro constraints simplifying significantly the string equations of motion can generally be solved in terms of two copies (left- and right-moving) of the Lorentz harmonic variables parameterizing the compact coset spaces isomorphic to the $(D-2)$-dimensional sphere [9]

$$S_{D-2} = \frac{SO(1,D-1)}{SO(1,1)\times SO(D-2)\times K_{D-2}} \tag{31}$$

Moreover, it was shown in [7] that from these variables the particular Lorentz covariant combinations can be formed which resolve generally the corresponding nonlinear σ-model equations of motion inspired by the bosonic strings in the geometrical approach [1]. By the construction the number of two copies of chiral variables parameterizing the coset space (31) is apparently enough to recover the $2(D-2)$ physical degrees of freedom of D-dimensional bosonic strings. But in the case of superstrings these variables replaced by the worldsheet superfields must be properly restricted to provide the necessary balance between bosonic and fermionic degrees of freedom $(D-2)B \equiv (D-2)F$.

In this Section we shall show that the suitable constraints can be achieved within the method of the nonlinear realization of superconformal symmetry developed in Section 1.2.

Proceeding from [7] one can check that the general solution of the Liouville Eq. (13) can be written in form

$$\begin{aligned} e^{-2\tilde{u}(\tilde{\xi}_L^{\pm\pm})} &= \frac{1}{2}\tilde{r}_m^{++}(\tilde{\xi}_L^{--})\tilde{l}^{--m}(\tilde{\xi}_L^{++}), \\ \tilde{m}_{++}^{--}(\tilde{\xi}_L^{++}) &= \tilde{l}_m^{--}(\tilde{\xi}_L^{++})\tilde{\partial}_{++}\tilde{l}^m(\tilde{\xi}_L^{++}), \\ \tilde{n}_{--}^{++}(\tilde{\xi}_L^{--}) &= \tilde{r}_m^{++}(\tilde{\xi}_L^{--})\tilde{\partial}_{--}\tilde{r}^m(\tilde{\xi}_L^{--}), \end{aligned} \tag{32}$$

where the left(right)-moving Lorentz harmonics are normalized as follows

$$\tilde{l}_m^{++}\tilde{l}^{m++} = 0, \quad \tilde{l}_m^{--}\tilde{l}^{m--} = 0, \quad \tilde{l}_m\tilde{l}^{m\pm\pm} = 0, \tag{33}$$

$$\tilde{l}_m^{--}\tilde{l}^{m++} = 2, \quad \tilde{l}_m\tilde{l}^m = -1. \tag{34}$$

Substituting these solutions into the Eqs. (29) and taking account of the expressions (25) one finds

$$M \equiv M^{--}(\xi_L^{++},\eta^+,\overline{\eta}^+) = (\overline{\tilde{D}}_+\overline{\eta}^+)\tilde{\eta}^+\overline{\tilde{\eta}}^+\tilde{l}_m^{--}(\tilde{\xi}_L^{++})\tilde{\partial}_{++}\tilde{l}^m(\tilde{\xi}_L^{++}), \tag{35}$$

$$N \equiv M^{++}(\xi_L^{--},\eta^-,\overline{\eta}^-) = (\overline{\tilde{D}}_-\overline{\eta}^-)\tilde{\eta}^-\overline{\tilde{\eta}}^-\tilde{r}_m^{++}(\tilde{\xi}_L^{--})\tilde{\partial}_{--}\tilde{r}^m(\tilde{\xi}_L^{--}) \tag{36}$$

Now comparing these SFs with explicit form of the general solution of the Eq. (6)

$$
\begin{aligned}
e^{-2W(\xi_L^{\pm\pm},\eta^{\pm})} &= \frac{1}{2} r_m^{++}(\xi_L^{--},\eta^{-}) l^{--m}(\xi_L^{++},\eta^{+}), \\
\Psi_{+}^{--}(\xi_L^{++},\eta^{+},\overline{\eta}^{+}) &= \frac{1}{2i} l_m^{--}(\xi_L^{++},\eta^{+}) D_{+} l^{m}(\xi_L^{++},\eta^{+}), \\
\Psi_{-}^{++}(\xi_L^{--},\eta^{-},\overline{\eta}^{-}) &= \frac{1}{2i} r_m^{++}(\xi_L^{--},\eta^{-}) D_{-} r^{m}(\xi_L^{--},\eta^{-}),
\end{aligned}
\tag{37}
$$

one can derives the following expressions for the corresponding SFs of linear realization

$$
M^{\pm\pm} = \Phi_{\pm}^{\pm}\Omega^{\mp}\Psi_{\mp}^{\pm\pm}, \qquad \overline{D}_{\pm}\Phi_{\pm}^{\pm} = 0, \qquad D_{\pm}\Omega^{\pm} = 1, \tag{38}
$$

$$
\Phi_{\pm}^{\pm} = \overline{\tilde{D}}_{\pm}\overline{\eta}^{\pm}, \qquad \Omega^{\pm} = (\tilde{D}_{\pm}\eta^{\pm})\tilde{\eta}^{\pm}, \tag{39}
$$

$$
\begin{aligned}
l_m^{\pm\pm,0}(\xi_L^{++},\eta^{+}) &= \tilde{l}_m^{\pm\pm,0}(\tilde{\xi}_L^{++}), \\
r_m^{\pm\pm,0}(\xi_L^{--},\eta^{-}) &= \tilde{l}_m^{\pm\pm,0}(\tilde{\xi}_L^{--}).
\end{aligned}
\tag{40}
$$

4 Conclusion

Thus, we have established that the $n=(2,2)$ generalization of complex Liouville equation appropriated to the $N=2, D=4$ superstring is given by the Eq. (6) in which the auxiliary SFs Ψ are subjected to the constraints (10) and (38). Then the general solution of this equation can be given in terms of Lorentz harmonics (37) which in one's turn are also restricted by the conditions (40). Note, that in its own rights this fact actually means that the Eq. (6) proved to be exactly solvable as the corresponding bosonic string equation does [7] but unlike to the bosonic case the corresponding harmonic SFs becomes *essentially restricted* by the constraints (40) which provide the supersymmetric balance between bosonic and fermionic degrees of freedom. [e] It is worth mentioning that the first constraint in Eqs. (38) implies that the SFs M are actually nilpotence $M^2=0$. From the theory of spontaneously broken supersymmetries we know that such a type of constraints leads directly to the nonlinear realizations of the underline symmetries [10], [11], in frame of which these constraints could be solved explicitly in terms of the corresponding Goldstone (super)fields. In the case under consideration we find the suitable manifestly supercovariant

[e] It is obvious, that the same property of $n=(1,1)$ super-Liouville equation [3] can be derived from this one with the help of dimensional reduction from (37) and (40).

solution (39) in terms of the Goldstone fermions of the nonlinear realization of $n = (2,2)$ superconformal symmetry $\lambda^{\pm}(\xi_L^{\pm\pm})$, $\overline{\lambda}^{\pm}(\xi_L^{\pm\pm})$.

We are convinced that this approach actually gives the universal way of deriving the equations of motion as well as their solutions for the superstrings in the cases of higher dimensions too, i.e. $D = 6, 10$. In particular, the $N = 2, D = 6$ superstring is expected should be described by the nonlinear realization of the $n = (4,4)$ supersymmetric WZNW σ-model in which W is replaced by the double-analytical SF $q^{(1,1)}$ representing twisted multiplet in the harmonic $(4,4)$ superspace [4], [12].

We hope return to this question in a forthcoming publications.

Acknowledgments

It is a great pleasure for me to express grateful to I. Bandos, E. Ivanov, S. Krivonos, A. Pashnev and D. Sorokin for interest to this work and valuable discussions.

References

1. F. Lund and T. Regge *Phys. Rev.* **D 14** 1524 (1976);
   R. Omnes *Nucl. Phys.* **B 149** 269 (1979);
   B.M. Barbashov and V.V. Nesterenko *Commun. Math. Phys.* **78** 499 (1981).
   A. Zheltukhin *Sov. J. Nucl. Phys. (Yadern.Fiz.)* **33** 1723 (1981); *Theor. Mat. Phys.* **52** 73 (1982); *Phys. Lett.* **B 116** 147 (1982); *Theor. Mat. Phys.* **56** 230 (1983).
2. I. Bandos, P. Pasti, D. Sorokin, M. Tonin and D. Volkov, *Nucl. Phys.* **B 446** 79-119 (1995)(**hep-th/9501113**).
3. I. Bandos, D. Sorokin and D. Volkov, *Phys. Lett.* **B 372** 77-82 (1996).
4. E. Ivanov and S. Krivonos, *J. Physics* **A17** L671 (1984).
5. E. Ivanov and A. Kapustnikov, *Nucl. Phys.* **B 333** 439 (1990);
6. A.S. Galperin, P.S. Howe and K.S. Stelle, *Nucl. Phys.* **B 368** 281 (1992).
7. I. Bandos, E. Ivanov, A. Kapustnikov and S. Ulanov, *J. Math. Phys.* **40** 5203-5223 (1999)(**hep-th/9810038**).
8. S. Coleman, J. Wess and B. Zumino, *Phys. Rev.* **177** 2239 (1969);
   C. Callan, S. Coleman, J. Wess and B. Zumino, *Phys. Rev.* **177** 2247 (1969);
   D. Volkov, *J. Elem. Part. Atom. Nucl.* **4** 3 (1973);
   V. Ogievtsky, *in Proc. X Winter School of Theor. Physics* (Wroclaw, 1974), **vol. 1**, p. 117.

9. D.B. Fairlie and C.A. Manogue, *Phys. Rev.* **D 36** 475-479 (1987); N. Berkovits, *Phys. Lett.* **B 241** 497-502 (1990).
10. E. Ivanov, A.A. Kapustnikov, *J. Physics* **A11** 2375 (1978); *J. Physics* **G8** 167 (1982); *Int. J. Mod. Phys.* **A 7** 2153 (1992).
11. M. Rocek, A.A. Tseytlin, **ITP-SB-98-68, Imperial/TP/98-99/013**, **hep-th/9811232**.
12. E. Ivanov, S. Krivonos and V. Leviant, *Nucl. Phys.* **B 304** 601 (1988);
    O. Gorovoy and E. Ivanov, *Nucl. Phys.* **B 381** 394 (1992);
    E. Ivanov and A. Sutulin, *Nucl. Phys.* **B 432** 246-280 (1994); *Nucl. Phys.* **B 483** 531 (1997)(E) ;
    E. Ivanov and A. Sutulin, *Class. Quantum Grav.* **14** 843-863 (1997).